ON THE EQUATIONS OF STEADY-STATE CONVECTION

(OB URAVNENIAKH STATSIONARNOI KONVEKTSII)

PMM Vol.27, No.2, 1963, pp. 295-300

M.R. UKHOVSKII and V.I. IUDOVICH
(Rostov-on-Don)

(Received December 18, 1962)

1. It is known that a heated fluid can be in equilibrium in a gravitational field only if the temperature distribution $T(x)$ in it has the form

$$T = T_0(x_3) = Cx_3 + C_0 \qquad (1.1)$$

where C and C_0 denote constants, the axis x_3 being directed vertically downwards. However, the preceding solution can turn out to be unstable: for example, a stable steady flow of the fluid can come into being. This is the case which we shall examine in the present paper. The flows of the type just mentioned, called steady-state natural convection, are described by the system of equations [1]

$$\nu\triangle\mathbf{v}' = (\mathbf{v}'\cdot\nabla)\,\mathbf{v}' + \nabla p' + \beta\mathbf{g}T', \quad \chi\triangle T' = \mathbf{v}'\nabla T', \quad \operatorname{div}\mathbf{v}' = 0 \qquad (1.2)$$

where the following notation is used: $\mathbf{v}'(x)$ is the velocity of the fluid; $x = (x_1, x_2, x_3)$ is a point in the three-dimensional space; $T'(x)$ is the temperature; $p'(x)$ the pressure; ν, χ, β, respectively, are the viscosity, thermal conductivity and thermal expansion; $\mathbf{g}(0, 0, g)$ is the acceleration due to gravity; the density of the fluid has been set equal to unity.

We shall seek a solution $(\mathbf{v}', p', T')$ of the system (1.2) in a bounded domain Ω which satisfies at its boundary S the set of conditions

$$\mathbf{v}'|_S = 0, \qquad T'|_S = Cx_3 + C_0 \qquad (1.3)$$

It is evident that

$$\mathbf{v}_0 = 0, \quad T_0 = Cx_3 + C_0, \quad p_0 = -\beta g\left(\tfrac{1}{2}Cx_3^2 + C_0x_3\right) + \text{const} \qquad (1.4)$$

solves the problem (1.2), (1.3). Performing the change of variables

$$\mathbf{v}' = \mathbf{v}_0 + \mathbf{v}, \qquad p' = p_0 + p, \qquad T' = T_0 + T \tag{1.5}$$

in (1.2) and (1.3), we have

$$\nu\triangle\mathbf{v} = (\mathbf{v}\cdot\nabla)\,\mathbf{v} + \nabla p + \beta\mathbf{g}T, \qquad \chi\Delta T = \mathbf{v}\nabla T + Cv_3, \qquad \operatorname{div}\mathbf{v} = 0 \tag{1.6}$$

$$v\,|_S = 0, \qquad T\,|_S = 0 \tag{1.7}$$

Together with the preceding problem, we shall consider the linearized problem

$$\nu\triangle\mathbf{v} = \nabla p + \beta\mathbf{g}T, \quad \chi\triangle T = Cv_3, \quad \operatorname{div}\mathbf{v} = 0; \qquad \mathbf{v}\,|_S = 0, \qquad T\,|_S = 0 \tag{1.8}$$

We shall ask for those values of the parameter C which lead to non-trivial solutions of the problem (1.6) and (1.7). In order to do this we shall reduce the problem (1.6) and (1.7) to the operator equation

$$\mathbf{v} = K\,(\mathbf{v}\ \ C) \tag{1.9}$$

in some Hilbert space H_1. The operator K is continuous, and this gives us the opportunity to apply the general theory of the bifurcation of solutions [2] to the study of equation (1.9). The system (1.8) reduces to the operator equation

$$\mathbf{v} = CB\mathbf{v} \tag{1.10}$$

where B, a linear operator, is the Frechet differential of operator K. Furthermore, the following acts can be established.

Theorem 1.1. The operator B is selfadjoint, positive and completely continuous. It follows that there exists a denumerable set of characteristic values of equation (1.10) (problem (1.8))

$$0 < C_1 < C_2, \ldots, C_n \to +\infty,$$

and a corresponding complete system of eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, ...

Theorem 1.2. (1) Equation (1.6) (problem (1.9), (1.7)) with $C < C_1$ possesses only a trivial solution. (2) The points of bifurcation can only be C_1, C_2, ... (3) Every number $C_k (k = 1, 2, \ldots)$ to which there corresponds a non-denumerable set of characteristic vectors of equation (1.10) constitutes a genuine point of bifurcation of equation (1.10). Section 4 of this paper contains an example of the application of Theorems 1.1 and 1.2.

Sorokin [3,4] devotes to the study of convection in a viscous fluid. Reference [3] establishes a variational principle which is satisfied by the eigenvectors and eigennumbers of problem (1.18), and a Ritz method for their approximate calculation is proposed. However, no proof of the existence of the eigenvalues has been given. In the process of proving Theorem 1.1 we derive an alternative variational principle (and an entirely different variant of Ritz's method). The justification of the variational principle and of the Ritz method given in [3] can be carried out by a method which is closely related to that given here. A proof (1) of Theorem 1.2 is given in [4]. In addition, it is proved that uniqueness occurs also for $C = C_1$, but the proof contains an error*. In spite of this, [4] establishes formal expansions of the solutions to the problem of convection into series of a special kind.

2. In order to derive the operator equations, we shall adopt a method similar to that developed in [5]. We define a Hilbert space H_1 as a closure of a set of smooth solenoidal vectors in space Ω which vanish near the boundary S of a norm given by the scalar

$$(\bar{u}, \mathbf{v})_{H_1} = \int_{\Omega} \operatorname{rot} \mathbf{u} \cdot \operatorname{rot} \mathbf{v} \, dx \tag{2.1}$$

Let H_2 be the closure of a set of functions smooth in the domain Ω which are equal to zero on the boundary, of a norm given by the scalar product

$$(f, g)_{H_2} = \int_{\Omega} \nabla f \cdot \nabla g \, dx \tag{2.2}$$

We shall call the pair $(\mathbf{v}, T)$, $\mathbf{v} \in H_1$, $T \in H_2$, which satisfies the integral identities

$$\nu (\mathbf{v}, \Phi)_{H_1} = -\int_{\Omega} (\mathbf{v}, \nabla) \mathbf{v} \Phi \, dx - \beta \int_{\Omega} T \mathbf{g} \, \Phi \, dx \tag{2.3}$$

$$\chi (T, \varphi)_{H_2} = -\int_{\Omega} \mathbf{v} \nabla T \varphi \, dx - C \int_{\Omega} v_3 \varphi \, dx \tag{2.4}$$

a generalized solution of problem (1.6), (1.7). Here $\Phi \in H_1$, $\varphi \in H_2$, and are arbitrary.

From the results in [6] it is easy to prove that the above generalized solution is infinitely differentiable everywhere inside Ω and satisfies equations (1.6) to (1.7); if the boundary of S is sufficiently

* In (2.10) and (1.5) (see [4, p. 199]) the symbol P refers to different quantities. For this reason on the right-hand side of (2.11) it is necessary to put ∇q and the subsequent statements lose their validity. The result itself is, evidently, correct.

smooth, then derivatives of an arbitrary, prescribed order will be continuous* in the closed domain Ω.

Lemma 2.1. It follows from (2.4) that

$$T = CA\mathbf{v} \tag{2.5}$$

where A denotes an operator operating from H_1 into H_2 which is uniformly continuous and independent of C. For an arbitrary function $f \in L_{6/5}(\Omega)$ we define an operator Lf by the integral identity

$$\chi (Lf, \varphi)_{H_2} = \int_\Omega f\varphi \, dx \tag{2.6}$$

for all $\varphi \in H_2$. Applying Sobolev's composition theorem [8], we find that the right-hand side of (2.6) represents a linear functional with regard to φ in H_2. It follows from Riesz' theorem on the general form of a linear functional in a Hilbert space that the operator L is determined; L denotes a linear operator from $L_p(\Omega)$ $(p \geqslant 6/5)$ in H_2 which is completely continuous for $p > 6/5$. It follows from (2.4) that

$$T = L_1 T = L_\mathbf{v} T - CLv_3, \qquad L_\mathbf{v} T = -L(\mathbf{v}\nabla T) \tag{2.7}$$

For a fixed $\mathbf{v} \in H_1$ the operator L_v is completely continuous in H_2. We shall prove that for $C = 0$ (2.7) implies $T = 0$. Indeed, multiplying (2.7) scalarly by T and applying (2.6) we find at once that $\| T \|_{H_2} = 0$. According to Fredholm's theorem it follows further that the inverse of the operator $E - L_v$ exists, and (2.5) is satisfied for

$$A\mathbf{v} = -(E - L_v)^{-1} L v_3$$

Let now $\mathbf{v}_n \to \mathbf{v}$ in H_1 for $n \to \infty$. We assume $T_n = CA\mathbf{v}_n$ and prove that $T_n \Rightarrow T = CA\mathbf{v}$ in H_2. We write (2.4) for T_m, $\mathbf{v}_m$ and T_n, $\mathbf{v}_n$. Evaluating these relations one from the other and substituting $\varphi = T_m - T_n$ we obtain

$$\chi \| T_m - T_n \|^2_{H_2} = \int_\Omega (\mathbf{v}_n - \mathbf{v}_m) \nabla T_m (T_m - T_n) \, dx -$$
$$- C \int_\Omega (v_{m3} - v_{n3})(T_m - T_n) \, dx \tag{2.8}$$

---

* Applying the methods of [5,6], it is possible to prove that a more general problem involving two nonhomogeneous equations (1.6) can be solved "in the whole" (see also [7]).

Estimating the right-hand side of (2.8) with the aid of Hölder inequality and the composition theorem, we obtain

$$\chi \| T_m - T_n \|_{H_2} \leqslant M \left(\| \mathbf{v}_m - \mathbf{v}_n \|_{L_4} \cdot \| A\mathbf{v}_m \|_{H_2} + C \| \mathbf{v}_m - \mathbf{v}_n \|_{L_2} \right) \quad (2.9)$$

where M is a constant* which depends only on domain Ω. Now, from (2.4) with $\varphi = T$, we can easily find that

$$\| A\mathbf{v} \|_{H_2} \leqslant CM_1 \| \mathbf{v} \|_{L_2} \quad (2.10)$$

Making use of the property of complete continuity of the composition of H_1 into L_p ($p < 6$), we find that

$$\| T_m - T_n \|_{H_2} \to 0 \quad \text{for } m, n \to \infty \quad (2.11)$$

It follows that the continuity of the operator A is uniform. The lemma is thereby proved.

We now define an operator $K_1\mathbf{F}$ for an arbitrary vector $\mathbf{F}(x) \in L_p(\Omega)$ ($p > 6/5$) through the identity

$$\nu (K_1 \mathbf{F}\, \mathbf{\Phi})_{H_1} = \int_\Omega \mathbf{F\Phi}\, dx \qquad (\mathbf{\Phi} \in H_1) \quad (2.12)$$

In the same way as was done earlier in relation to operator L it is possible to establish that K_1 is a linear and completely continuous operator.

Lemma 2.2. Every generalized solution ($\mathbf{v}$, T) of problem (1.6), (1.7) in the sense of (2.3), (2.4) satisfies the operator equations

$$\mathbf{v} = K(\mathbf{v}, C), \qquad K(\mathbf{v}, C) = K_2\mathbf{v} + CK_3\mathbf{v} \quad (2.13)$$

$$K_2\mathbf{v} = -K_1 (\mathbf{v}, \nabla)\mathbf{v}, \qquad K_3\mathbf{v} = -K_1 (\beta g A\mathbf{v}) \quad (2.14)$$

where K_2 and K_3 are completely continuous in H_1. Conversely, every solution ($\mathbf{v}$, T) of the system (2.13) to (2.14) constitutes a generalized solution.

The justification for Lemma 2.2 follows immediately from Lemma 2.1 and the property of complete continuity of operator K_1.

---

* In what follows, M, M_i denote consecutive constants whose values depend exclusively on the domain Ω.

Let us first determine the generalized solution of the linearized problem (1.8) in a form of the pair $(\mathbf{v}, T)$, $\mathbf{v} \in H_1$, $T \in H_2$ which satisfies the integral identities

$$\nu (\mathbf{v}, \Phi)_{H_1} = -\beta \int_\Omega T \mathbf{g}\, \Phi \, dx, \quad \chi (T, \varphi)_{H_2} = -C \int_\Omega v_3 \varphi \, dx \qquad (2.15)$$
$$(\Phi \in H_1) \qquad\qquad (\varphi \in H_2)$$

Lemma 2.3. The quest for the generalized solutions of problem (1.8) in the sense of (2.15) is equivalent to the solution of the operator equation

$$\mathbf{v} = CK_1 (\beta \mathbf{g} L v_3) \qquad (2.16)$$

where the operator on the right-hand side is completely continuous and constitutes a Fréchet differential of operator $K(\mathbf{v}, C)$ at the point $\mathbf{v} = 0$. The proof of this lemma follows from the determination of operators K_1, L and some simple estimates in an obvious way which we shall omit for the sake of brevity.

3. In order to prove Theorem 1.1, we observe that in view of Lemmas 2.2 and 2.3, it is sufficient to prove that operator $B\mathbf{v} = K_1(\beta \mathbf{g} L v_3)$ is self-adjoint and positive in H_1.

The fact that B is selfadjoint is a consequence of (2.6) and (2.12) and of the following chain of equations which are satisfied for arbitrary $\mathbf{v}$, $\mathbf{w} \in H_1$

$$(B\mathbf{v}, \mathbf{w})_{H_1} = (K_1 (\beta \mathbf{g} L v_3), \mathbf{w})_{H_1} = \frac{1}{\nu} \int_\Omega \beta \mathbf{g} L v_3 \cdot \mathbf{w} \, dx =$$
$$= \frac{\beta g}{\nu} \int_\Omega L v_3 w_3 \, dx = \frac{\beta g \chi}{\nu} (L v_3, L w_3)_{H_2} = (\mathbf{v}, B\mathbf{w})_{H_1} \qquad (3.1)$$

It follows from (3.1) with the substitution $\mathbf{w} = \mathbf{v}$ that

$$(B\mathbf{v}, \mathbf{v})_{H_1} = \frac{\beta g \chi}{\nu} \| L v_3 \|_{H_2}^2 \geqslant 0 \qquad (3.2)$$

This means that the operator is positive. Theorem 1.1 has been proved.

Let C_1 denote the smallest eigenvalue of the linear problem (1.8), and let $(\mathbf{v}_1, T_1, p_1)$ be its corresponding characteristic solution. Multiplying the first equation (1.8) by $\mathbf{v}_1$, and the second equation (1.8) by T_1 and integrating, we obtain

$$\nu \| \mathbf{v}_1 \|_{H_1}^2 + \chi \| T_1 \|_{H_2}^2 + (\beta g + C_1) \int_\Omega v_{13} T_1 \, dx = 0 \qquad (3.3)$$

On the other hand, as shown in [4], equations (1.8) will constitute the Euler equations of the variational problem

$$J(\mathbf{v}, T) = \frac{1}{2}[\nu\|\mathbf{v}\|^2_{H_1} + \chi\|T\|^2_{H_2}] \tag{3.4}$$

$$-\int_\Omega v_3 T\,dx = 1, \qquad \operatorname{div}\mathbf{v} = 0 \tag{3.5}$$

This means that if there exist values $\mathbf{v} = \mathbf{v}_+$, $T = T_+$ which render the functional (3.4) a minimum subject to the conditions (3.5), then in conjunction with some $P = P_+$ they will constitute the solution of problem (1.8) for some $C = C_+$. It is clear that

$$\beta g + C_+ = \frac{\nu\|\mathbf{v}_+\|^2_{H_1} + \chi\|T_+\|^2_{H_2}}{-\int_\Omega v_{3+}T_+dx} = \min \frac{\nu\|\mathbf{v}\|^2_{H_1} + \chi\|T\|^2_{H_2}}{-\int_\Omega v_3 T\,dx} \tag{3.6}$$

Taking into account (3.3) and the fact that C_1 is a minimum, equation (3.6) leads to

$$C_1 = C_+ = \min \frac{\nu\|\mathbf{v}\|^2_{H_1} + \chi\|T\|^2_{H_2}}{-\int_\Omega v_3 T\,dx} - \beta g \tag{3.7}$$

Let first $(\mathbf{v}, p, T)$ be an arbitrary, non-trivial solution of the non-linear problem (1.6) which corresponds to some C. Proceeding exactly in the same way as in connection with the derivation of (3.3), we obtain

$$C = \frac{\nu\|\mathbf{v}\|^2_{H_1} + \chi\|T\|^2_{H_2}}{-\int_\Omega v_3 T\,dx} - \beta g \tag{3.8}$$

Finally, from (3.7) and (3.8), we obtain that $C_1 \leqslant C$.

The validity of Sections 2 and 3 follows from the results obtained by Krasnoselskii [2] on the basis of Lemmas 2.2 and 2.3. The theorem has been proved completely.

4. In general it is impossible to establish the multiplicity of the eigenvalues of problem (1.8), i.e. in view of the self-adjointness of operator B, the number of characteristic functions which corresponds to a given eigenvalue cannot be determined.

We now adduce an example for which this can be done, and for which the preceding considerations lead to the demonstration of the existence of bifurcation points.

We shall seek solutions $\mathbf{v}$, T, p of system (1.6) which are periodic in x_1, x_2 and x_3 with periods 2α, 2δ and 2γ which render v_1 denumerable with respect to x_2, x_3 and non-denumerable with respect to x_1; v_2 is denumerable with respect to x_2, x_3 and non-denumerable with respect to x_3; v_3, T are denumerable with respect to x_1, x_2 and non-denumerable with respect to x_3, p is denumerable with respect to x_1, x_2, x_3. It is not difficult to show that Theorems 1.1 and 1.2 are satisfied also in this case because the requirement of periodicity completely replaces the preceding boundary conditions.

We shall now remark further that the corresponding linearized problem reduces itself to the determination of T from the equation

$$\Delta^3 T = \frac{\beta g C}{\varkappa \nu}(T_{x_1x_1} + T_{x_2x_2}) \tag{4.1}$$

if $\mathbf{v}$ and p are excluded, subject to the preceding conditions of periodicity and denumerability. The eigenvalues of this problem are

$$\lambda_{kmn} = \frac{\nu \chi \pi^4}{\beta g}\left(\frac{k^2}{\alpha^2} + \frac{m^2}{\delta^2} + \frac{n^2}{\gamma^2}\right)^3 \left(\frac{k^2}{\alpha^2} + \frac{m^2}{\delta^2}\right)^{-1} \tag{4.2}$$
$$(k^2 + m^2 \neq 0;\ k,\ m = 0, 1, 2, \ldots;\ n = 1, 2, \ldots)$$

to which there correspond the characteristic functions

$$T_{kmn} \doteq \cos\frac{k\pi x_1}{\alpha}\cos\frac{m\pi x_2}{\delta}\sin\frac{n\pi x_3}{\gamma} \tag{4.3}$$

We remark that $\lambda_{kmn} \geqslant \nu\chi\pi^4/\beta g = \lambda_0$, and this means that for $C < \lambda_0$, the preceding solution of problem (1.6) does not exist for any values of α, δ and γ.

If to the eigenvalue λ_{kmn} there were to correspond more than one characteristic function, then $\lambda_{kmn} = \lambda_{k_1m_1n_1}$ for some other choice of k_1, m_1, n_1. Let us fix α and δ. Then γ can be uniquely expressed in terms of k, m, n, k_1, m_1, n_1, α and δ. It is clear that the multicipliсity of such gammas as correspond to all possible choices of k, m, n, k_1, m_1 and n_1 cannot be higher than denumerable. Hence, for arbitrary values γ, except perhaps for some denumerable set, all eigenvalues will be simple and each of them will constitute a point of bifurcation.

BIBLIOGRAPHY

1. Landau, L.D. and Lifshits, E.M., *Mekhanika sploshnykh sred (Mechanics of Continua)*. GITTL, 1954.

2. Krasnoselskii, M.A., *Topoligicheskie metody v teorii nelineinykh integral'nykh uravnenii (Topological Methods in the Theory of Nonlinear Integral Equations)*. GITTL, 1956.

3. Sorokin, V.S., Variatsionnyi metod v teorii konvektsii (Variational method in convection). *PMM* Vol. 17, No. 1, 1953.

4. Sorokin, V.S., O statsionarnykh dvizheniiakh v zhidkosti, podogrevaemoi snizu (On steady-state motions of a fluid heated from below). *PMM* Vol. 18, No. 2, 1954.

5. Vorovich, I.I. and Iudovich, V.I., Statsionarnoe techenie viazkoi zhidkosti (Steady motion of a viscous fluid). *Dokl. Akad. Nauk SSSR* Vol. 124, No. 2, 1959.

6. Vorovich, I.I. and Iudovich, V.I., Statsionarnoe techenie viazkoi neszhimaemoi zhidkosti (Steady motion of a viscous incompressible fluid). *Matem. sb.* Vol. 53, No. 4, 1961.

7. Ladyzhenskaia, O.A., *Matematicheskie voprosy dinamiki viazkoi neszhimaemoi zhidkosti (Mathematical Problems of the Dynamics of Viscous Incompressible Flows)*. Fizmatgiz, 1961.

8. Sobolev, S.L., *Nekotorye primeneniia funktsional'nogo analiza v matematicheskoi fizike (Some Applications of Functional Analysis in Mathematical Physics)*. Izd-vo Leningr. Univers., 1950.

Translated by J.K.